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# Derivation of some inequalities using growth indicators of entire Algebroidal functions

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## Abstract

In this paper some inequalities representing the growth estimates of entire algebroidal functions using the tools of relative growth indicators such as relative type and relative weak type of entire algebroidal functions have been established.In fact the inequalities obtained here are sharper in the sense of estimation.

AMS Subject Classification (2010) : 30D30,30D35.

***Keywords:***

Entire algebroidal function, Growth estimates, Order(lower order), Relative growth indicator, Slowly changing function.

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## 1. Introduction

Let $f_i\ (i = 0,1,2,\ldots,k)$ be entire functions such that they have no common zeros and $f_k \neq 0$. We define a k-valued function F with the following irreducible equation constructed by $f_i\ (i = 0,1,2,\ldots,k)$ with $f_k \neq 0$:

$$f_k F^K + f_{k-1} F^{k-1} + f_{k-2} F^{k-2} + \ldots\ldots\ldots\ldots + f_0 = 0.$$

We can say F as a k-valued algebroidal function if at least one of the $f_i\ (i = 0,1,2,\ldots,k)$ is transcendental.Moreover, for $f_k \equiv 1$ F is defined to be a k-valued entire algebroidal function.

Bernal[2] coined the notion of relativeness of growth indicators of entire functions and following this concept, several researchers like Datta et al.[4] established more powerful results taking into consideration of some advanced class of growth indicators one of which is from the view point of slowly changing function L(r) which is nothing but a positive continuous function satisfying

$$\frac{L(ar)}{L(r)} \to 1 \ as \ r \to \infty,$$

where 'a' being a positive constant. In this connection the notion of index pair in case of entire functions may be recalled :

Juneja, Kapoor and Bajpai [7] defined the (p,q)-th order $\rho_f(p,q)$ and (p,q)-th lower order $\lambda_f(p,q)$ of an entire function f respectively as follows:

$$\rho_f(p,q) = \lim_{r\to\infty} \sup \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \lim_{r\to\infty} \inf \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}$$

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where p,q are any two positive integers with $p \geq q$. These definitions extend the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f considered by Titchmarsh[12] for each integer l≥2 since these correspond to the particular case $\rho_f^{[l]} = \rho_f(l,1)$ and $\lambda_f^{[l]} = \lambda_f(l,1)$ . Clearly, $\rho_f(2,1) = \rho_f$ and $\lambda_f(2,1) = \lambda_f$.

In this connection, let us recall that if $0 < \rho_f(p,q) < \infty$, then the following properties hold

$$\rho_f(p-n,q) = \infty \; for \; n < p, \rho_f(p,q-n) = 0 \; for \; n < q, and \; \rho_f(p+n,q+n) = 1 \; for \; n = 1,2,..$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \; for \; n < p, \lambda_f(p,q-n) = 0 \; for \; n < q, and \; \lambda_f(p+n,q+n) = 1 \; for \; n = 1,2, \ldots .$$

Recalling that for any pair of integers m and n the Kronecker function is defined by

$$\delta_{m,n} = 1 \;\; for \;\; m = n \;\; and$$
$$\delta_{m,n} = 0 \;\; for \;\; m \neq n,$$

the aforementioned properties provide the following definitions:

**1 Definition**:*[7]* An entire function f is said to have index-pair (1,1) if $0 < \rho_f(1,1) < \infty$. Otherwise, f is said to have index-pair $(p,q) \neq (1,1), p \geq q \geq 1$, if $\delta_{p-q,0} < \rho_f(p,q) < \infty \; and \; \rho_f(p-1,q-1) \notin \mathbb{R}^+$.

**2 Definition**: *[7]* An entire function f is said to have lower index-pair (1,1) if $0 < \lambda_f(1,1) < \infty$.. Otherwise, f is said to have lower index-pair $(p,q) \neq (1,1), p \geq q \geq 1, \delta_{p-q,0} < \lambda_f(p,q) < \infty \; and \; \lambda_f(p-1,q-1) \notin \mathbb{R}^+$.

An entire function f of index-pair *(p,q)* is said to be of regular *(p,q)*-growth if its *(p,q)-th* order coincides with its *(p,q)-th* lower order, otherwise f is said to be of irregular *(p,q)*-growth.

Further, to compare the growths of entire functions having the same *(p,q)-th* order, Juneja, Kapoor and Bajpai [8] also introduced the concepts of *(p,q)-th* type and *(p,q)-th* lower type. Using the concept of relative growth indicators as initiated by Bernal [2] along with the help of the slowly changing functions Datta et al.[5] proved some results in the direction of relative L-order(relative L- lower order), relative L-type(relative L- weak type). In a like manner these can be developed from the view point of relative L-index pair of an entire function *f* with respect to the entire function *g* as follows:

$$(\boldsymbol{i}) \;\; relative\,(p,q)-th\,L-order \;\; {}^{L}\rho_g^{(p,q)} = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{(\log^{[q]}(rL(r))};$$

$$(ii) relative(p,q)-th\,L- \; lower\,order \;\; {}^{L}\lambda_g^{(p,q)} = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]}\big(rL(r)\big)};$$

$$(iii)\; relative\,(p,q)-th\,L-type \;\; {}^{L}\sigma_g^{(p,q)} = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\big(\log^{[q-1]}\big(rL(r)\big)\big)^{{}^{L}\rho_g^{(p,q)}(f)}};$$

$$(iv)\; relative(p,q)-th\,L- \; weak\,type \;\; {}^{L}\tau_g^{(p,q)} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\big(\log^{[q-1]}\big(rL(r)\big)\big)^{{}^{L}\lambda_g^{(p,q)}(f)}};$$

$$(v) relative\,(p,q)-th\,L \; -lower\,type \;\; {}^{L}\bar{\sigma}_g^{(p,q)} = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\big(\log^{[q-1]}\big(rL(r)\big)\big)^{{}^{L}\rho_g^{(p,q)}(f)}};$$

$$(vi) the\;growth\;indicator \;\; {}^{L}\bar{\tau}_g^{(p,q)} = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\big(\log^{[q-1]}(rL(r))\big)^{{}^{L}\lambda_g^{(p,q)}(f)}}.$$

In this paper we wish to derive some inequalities focusing on the comparative growth estimates of entire algebroidal functions by means of relative growth indicators especially different kinds of types of those functions from the view point of slowly changing functions. The standard definitions, theories and techniques related to our discussion will not be explained in detail as those are available in Valiron[13] and Holland[6].

## 2. Research Method : Lemmas:

In this section we will present some lemmas which will be needed in our subsequent discussion.

First of all let us recall the following theorem due to Ruiz et al. [10]

**Theorem A [10]** : Let f and g be any two entire functions with index-pair *(m,q) and (m,p)*, respectively, where p,q,m are all positive integers such that *m≥p and m≥q*. Then

$$\frac{\lambda_f(m,q)}{\rho_g(m,p)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\} \leq max\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_f(m,q)}{\lambda_g(m,p)}.$$

From the conclusion of the above theorem, we present the following two lemmas which will be needed in the sequel.

**Lemma 1 [10]** Let *f* be an entire function with index-pair *(m,q) and g* be an entire function of regular *(m,p)*-growth where *p,q,m* are all positive integers such that *m≥p and m≥q.* Then

$$\rho_g^{(p,q)}(\boldsymbol{f}) = \frac{\rho_f(m,q)}{\rho_g(m,p)} \;\; \boldsymbol{and} \; \lambda_g^{(p,q)}(\boldsymbol{f}) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)}.$$

**Lemma 2 [10]** Let *f* be an entire function with index-pair *(m,q) and g* be an entire function of regular *(m,p)*-growth where *p,q,m* are all positive integers such that *m≥p and m≥q.* Then

$$\rho_g^{(p,q)}(\boldsymbol{f}) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)} \text{ and } \lambda_g^{(p,q)}(\boldsymbol{f}) = \frac{\rho_f(m,q)}{\rho_g(m,p)}.$$

## 3. Results and Analysis.

Here we state the main results of the chapter. In this section let us first mention the primary assumptions in carrying out all the theorems along with the corollaries deduced from them.The assumptions are as follows:

Let *F* and *G* be two *k*-vauled entire algebroidal functions such that

$$F^k + f_{k-1}F^{k-1} + f_{k-2}F^{k-2} + \ldots\ldots + f_0 = 0$$
$$G^k + g_{k-1}G^{k-1} + g_{k-2}G^{k-2} + \ldots\ldots + g_0 = 0$$

where $f_i$ $(\mathrm{i} = 0,1,2,\ldots.,\mathrm{k}-1)$ and $g_i$ $(\mathrm{i} = 0,1,2,\ldots.,\mathrm{k}-1)$ are entire functions having no common zeros.Also let us assume that

(i) both F and G have relative L-index pair *(p,q)*,

(ii) $f_i$ $(\mathrm{i} = 0,1,2,\ldots.,\mathrm{k}-1)$are entire functions with relative index-pairs *(m,q)* ,

(iii) $g_i$ $(\mathrm{i} = 0,1,2,\ldots.,\mathrm{k}-1)$ are entire functions with relative index-pairs *(m,p)* and

(iv) $g_i$ 's are of regular *(m,p)*- growth

where *p,q,m* are all positive integers such that *m ≥ min{p,q}.*

Now we will present the detailed proof of Theorem 1 for the sake of completeness which in fact can be regarded the main theorem of the present paper. The others are basically omitted since they are easily proven with the same techniques or with some easy reasoning.

**Theorem 1** In connection to the primary assumptions as stated above we get the following conclusion:

$$\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}} \leq {}^{L}\bar{\sigma}_G^{(p,q)}(F) \leq min\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}\right\}$$

$$\leq \max\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right\}$$

$$\leq {}^{L}\sigma_G^{(p,q)}(F) \leq \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,q)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

where the notations have already been explained in Section 1.

**Proof** From the definitions of ${}^{L}\sigma_{f_i}(m,q)$ and ${}^{L}\bar{\sigma}_{f_i}(m,q)$, we have for all sufficiently large values of r that

$$M_{f_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}, \quad (1)$$

$$M_{f_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\} \quad (2)$$

and also for a sequence of values of r tending to infinity we get that

$$M_{f_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}, \quad (3)$$

$$M_{f_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\} \quad (4)$$

Similarly from the definitions of ${}^{L}\sigma_{g_i}(m,p)$ and ${}^{L}\bar{\sigma}_{f_i}(m,q)$ it follows for all sufficiently large values of r that

$$M_{g_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}$$

$$r \leq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e.,}\quad M_{g_i}{}^{-1}(r) \geq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right], \quad (5)$$

$$M_{g_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e.,}\quad r \leq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e.,}\quad M_{g_i}{}^{-1}(r) \geq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right] \quad (6)$$

and for a sequence of values of r tending to infinity we obtain that

$$M_{g_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e.,}\quad r \geq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e.,}\quad M_{g_i}{}^{-1}(r) \leq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right], \tag{7}$$

$$M_{g_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e.,}\quad r \leq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\rho_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e.,}\quad M_{g_i}{}^{-1}(r) \geq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right]. \tag{8}$$

Now from (3) and in view of (5), we get for a sequence of values of r tending to infinity that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}\right]$$

$$\text{i.e.,}\ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e.,}\quad log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e.,}\quad \frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \geq \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{9}$$

As ε(>0) is arbitrary, in view of Lemma 2 it follows that

$$\limsup_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \geq \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$${}^{L}\sigma_G^{(p,q)}(F) \geq \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{10}$$

Analogously from (2) and in view of (8), it follows for a sequence of values of r tending to infinity that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}\right]$$

$$\text{i.e.,}\ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e.,}\quad log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e.,}\quad \frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \geq \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{11}$$

Since ε(>0) is arbitrary, we get from above and Lemma 2 that

$$\limsup_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \geq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$${}^{L}\sigma_G^{(p,q)}(F) \geq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{12}$$

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right]$$

i.e., ., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \le \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$ (13)

Since ε(>0) is arbitrary, we obtain in view of Lemma 2 that

$$\limsup_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \le \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e, ${}^{L}\sigma_G^{(p,q)}(F) \le \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$ (14)

Again from (2) and in view of (5), we get for all sufficiently large values of r that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \ge \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$ (15)

As ε(>0) is arbitrary, it follows from above and Lemma 2 that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \ge \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e, ${}^{L}\bar{\sigma}_G^{(p,q)}(F) \ge \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$ (16)

Also in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}^{(m,q)}}\right)\right\}}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \le \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$ (17)

Since ε(>0) is arbitrary, we get from Lemma 1 and above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\leq\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e, $${}^{L}\bar{\sigma}_G^{(p,q)}(F)\leq\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{18}$$

Similarly from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that

$$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq\log^{[p-1]}M_{g_i}{}^{-1}\left[\exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right]$$

i.e., $$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq\log^{[p-1]}\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e., $$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq\left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e., $$\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}}\leq\left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{19}$$

As ε(>0) is arbitrary, we obtain from Lemma 1 and above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\leq\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

i.e, $${}^{L}\bar{\sigma}_G^{(p,q)}(F)\leq\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{20}$$

Thus the theorem follows from (10), (12), (14), (16), (18) and (20).

**Remark 1** Let us consider $m=1,p=q=1,i=0,k=1;F=\exp z,G=\exp z,h=\exp z,f_0=z,g_0=z$ and $L(r)=\log r$. Then we see that the equality cannot be removed in Theorem 1.

Under the assumptions of Theorem 1, the following corollary may easily be obtained :

**Corollary 1**

**(1)** ${}^{L}\sigma_G^{(p,q)}(F)=\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$ and ${}^{L}\bar{\sigma}_G^{(p,q)}(F)=\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$ .

(2) In addition, if ${}^{L}\sigma_{f_i}(m,q)={}^{L}\sigma_{g_i}(m,p)$. Then ${}^{L}\sigma_G^{(p,q)}(F)={}^{L}\bar{\sigma}_G^{(p,q)}(F)=1.$

(3) In addition, if ${}^{L}\sigma_{f_i}(m,q)={}^{L}\bar{\sigma}_{f_i}(m,q)$ . Then

$${}^{L}\sigma_G^{(p,q)}(F)={}^{L}\bar{\sigma}_G^{(p,q)}(F)=\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$$

(4) In addition to the following condition ${}^{L}\sigma_{f_i}(m,q)={}^{L}\sigma_{g_i}(m,p)$ ,this result follows:

$${}^{L}\sigma_G^{(p,q)}(F)={}^{L}\bar{\sigma}_G^{(p,q)}(F)={}^{L}\sigma_F^{(q,p)}(G)={}^{L}\bar{\sigma}_F^{(q,p)}(G)=1.$$

(5) (i) ${}^{L}\sigma_G^{(p,q)}(F)={}^{L}\bar{\sigma}_G^{(p,q)}(F)=\infty$ *when* ${}^{L}\sigma_{g_i}(m,p)=0$ and

(ii) ${}^{L}\sigma_G^{(p,q)}(F)={}^{L}\bar{\sigma}_G^{(p,q)}(F)=0$ *when* ${}^{L}\sigma_{g_i}(m,p)=\infty$.

(6) (i) ${}^{L}\sigma_G^{(p,q)}(F)=0$ *when* ${}^{L}\sigma_{f_i}(m,q)=0$ ,

(ii) ${}^{L}\bar{\sigma}_G^{(p,q)}(F)=0$ *when* ${}^{L}\bar{\sigma}_{f_i}(m,q)=0,$

(iii) ${}^{L}\sigma_G^{(p,q)}(F)=\infty$ *when* ${}^{L}\sigma_{f_i}(m,q)=\infty$ and

(iv) ${}^{L}\bar{\sigma}_G^{(p,q)}(F)=\infty$ *when* ${}^{L}\bar{\sigma}_{f_i}(m,q)=\infty$ .

In the line of Theorem 1 and with the help of Lemma 2 , one can easily establish the following theorem and therefore its proof is omitted:

**Theorem 2** Following the assumptions of Theorem 1, the result follows :

$$\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}} \leq {}^{L}\tau_G^{(p,q)}(F) \leq min\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right\}$$

$$\leq \max\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\right\} \leq {}^{L}\bar{\tau}_G^{(p,q)}(F) \leq \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$$

**Proof** From the definitions of ${}^{L}\bar{\tau}_{f_i}(m,q)$ and ${}^{L}\tau_{f_i}(m,q)$, we have for all sufficiently large values of r that

$$M_{f_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q) + \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}, \quad (21)$$

$$M_{f_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q) - \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}, \quad (22)$$

and also for a sequence of values of r tending to infinity we get that

$$M_{f_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q) - \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}, \quad (23)$$

$$M_{f_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q) + \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}. \quad (24)$$

Similarly from the definitions of ${}^{L}\bar{\tau}_{g_i}(m,q)$ and ${}^{L}\tau_{f_i}(m,q)$ it follows for all sufficiently large values of r that

$$M_{g_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{g_i}(m,p) + \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e., } r \leq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{g_i}(m,p) + \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e., } M_{g_i}{}^{-1}(r) \geq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\right], \quad (25)$$

$$M_{g_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\tau_{g_i}(m,p) - \epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e., } r \geq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\tau_{g_i}(m,p) - \epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e., } M_{g_i}{}^{-1}(r) \leq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right], \quad (26)$$

and for a sequence of values of r tending to infinity we obtain that

$$M_{g_i}(r) \geq exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{g_i}(m,p) - \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e., } r \geq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{g_i}(m,p) - \epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e., } M_{g_i}{}^{-1}(r) \leq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\bar{\tau}_{g_i}(m,q)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right], \quad (27)$$

$$M_{g_i}(r) \leq exp^{[m-1]}\left\{\left({}^{L}\tau_{g_i}(m,p) + \epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}$$

$$\text{i.e., } r \leq M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\tau_{g_i}(m,p) + \epsilon\right)\left(\left(log^{[p-1]}(rL(r))\right)^{{}^{L}\lambda_{g_i}(m,p)}\right)\right\}\right]$$

$$\text{i.e., } M_{g_i}{}^{-1}(r) \geq exp^{[p-1]}\left[\left(\frac{log^{[m-1]}(rL(r))}{\left({}^{L}\tau_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\right], \quad (28)$$

Now using the same technique of Theorem 1, one can easily prove the conclusion of the present theorem by the help of Lemma 2 .Therefore the remaining part of the proof of the present theorem is omitted.

In view of Theorem 2, the following corollary can easily be derived:

**Corollary 2**

**(1)** ${}^{L}\bar{\tau}_G^{(p,q)}(F) = \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$ and ${}^{L}\tau_G^{(p,q)}(F) = \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$.

(2) In addition, if ${}^{L}\sigma_{f_i}(m,q) = {}^{L}\bar{\sigma}_{g_i}(m,p)$. Then ${}^{L}\bar{\tau}_G^{(p,q)}(F) = {}^{L}\tau_F^{(q,p)}(G) = 1.$

(3) In addition, if ${}^{L}\sigma_{f_i}(m,q) = {}^{L}\bar{\sigma}_{f_i}(m,q)$ and $f_i$ are entire functions of regular (m,q)- growth . Then

$${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$$

(4) In addition to the following condition ${}^{L}\sigma_{f_i}(m,q) = {}^{L}\bar{\sigma}_{g_i}(m,p)$ and g is of regular (m,p)-th growth then

$${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = {}^{L}\bar{\tau}_{F}^{(q,p)}(G) = {}^{L}\tau_{F}^{(q,p)}(G) = 1.$$

(5) (i) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\sigma_{g_i}(m,p) = 0$ and

(ii) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\sigma_{g_i}(m,p) = \infty.$

(6) (i) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\sigma_{f_i}(m,q) = 0$ ,

(ii) ${}^{L}\tau_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\bar{\sigma}_{f_i}(m,q) = 0,$

(iii) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\sigma_{f_i}(m,q) = \infty$ and

(iv) ${}^{L}\tau_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\bar{\sigma}_{f_i}(m,q) = \infty$ .

As a consequence of Theorem 1 and Theorem 2 and with the help of Lemma 1 and Lemma 2, the following two theorems can be proved easily and therefore their proofs are omitted:

**Theorem 3** In connection to the assumptions of Theorem 1 we get the following conclusion:

$$\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}} \le {}^{L}\tau_{G}^{(p,q)}(F) \le min\left\{\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\}$$

$$\le \max\left\{\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\} \le {}^{L}\bar{\tau}_{G}^{(p,q)}(F) \le \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}.$$

In view of Theorem 3, the following corollary may also be obtained:

**Corollary 3**

**(1)** ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}$ and ${}^{L}\tau_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}$ .

(2) In addition, if ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\tau_{g_i}(m,p)$. Then ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{F}^{(q,p)}(G) = 1.$

(3) In addition, if ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\tau_{f_i}(m,q)$ and $f_i$ are entire functions of regular (m,q)- growth . Then

$${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}.$$

(4) In addition to the following condition ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\bar{\tau}_{g_i}(m,p)$ and $f_i$ 's are of regular (m,q)-th growth then

$${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = {}^{L}\bar{\tau}_{F}^{(q,p)}(G) = {}^{L}\tau_{F}^{(q,p)}(G) = 1.$$

(5) (i) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\bar{\tau}_{g_i}(m,p) = 0$ and

(ii) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\bar{\tau}_{g_i}(m,p) = \infty.$

(6) (i) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\bar{\tau}_{f_i}(m,q) = 0$ ,

(ii) ${}^{L}\tau_{G}^{(p,q)}(F) = 0 \ when \ {}^{L}\tau_{f_i}(m,q) = 0,$

(iii) ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\bar{\tau}_{f_i}(m,q) = \infty$ and

(iv) ${}^{L}\tau_{G}^{(p,q)}(F) = \infty \ when \ {}^{L}\tau_{f_i}(m,q) = \infty$ .

**Theorem 4** Under the primary assumptions as laid down in Theorem 1 the following conclusion can be derived:

$$\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}} \le {}^{L}\bar{\sigma}_{G}^{(p,q)}(F) \le min\left\{\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\}$$

$$\le \max\left\{\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\} \le {}^{L}\sigma_{G}^{(p,q)}(F) \le \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}.$$

The following remark can be highlighted from the view of Theorem 4 :

**Remark 2**

**(1)** ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}$ and ${}^{L}\tau_{G}^{(p,q)}(F) = \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}$ .

(2) In addition, if ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\bar{\tau}_{g_i}(m,p)$ and $f_i$ 's are of regular (m,q)− growth then ${}^{L}\bar{\tau}_{G}^{(p,q)}(F) = {}^{L}\tau_{F}^{(q,p)}(G) = 1.$

(3) In addition, if ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\tau_{f_i}(m,q)$ and $f_i$ are entire functions of regular (m,q)- growth . Then

$$ {}^{L}\bar{\tau}_G^{(p,q)}(F) = {}^{L}\tau_G^{(p,q)}(F) = \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}. $$

(4) In addition to the following condition ${}^{L}\bar{\tau}_{f_i}(m,q) = {}^{L}\bar{\tau}_{g_i}(m,p)$ and $f_i$ 's are of regular (m,q)-th growth then

$$ {}^{L}\bar{\tau}_G^{(p,q)}(F) = {}^{L}\tau_G^{(p,q)}(F) = {}^{L}\bar{\tau}_F^{(q,p)}(G) = {}^{L}\tau_F^{(q,p)}(G) = 1. $$

(5) (i) ${}^{L}\bar{\tau}_G^{(p,q)}(F) = {}^{L}\tau_G^{(p,q)}(F) = \infty$ *when* ${}^{L}\bar{\tau}_{g_i}(m,p) = 0$ and

(ii) ${}^{L}\bar{\tau}_G^{(p,q)}(F) = {}^{L}\tau_G^{(p,q)}(F) = 0$ *when* ${}^{L}\bar{\tau}_{g_i}(m,p) = \infty$.

(6) (i) ${}^{L}\bar{\tau}_G^{(p,q)}(F) = 0$ *when* ${}^{L}\bar{\tau}_{f_i}(m,q) = 0$ ,

(ii) ${}^{L}\tau_G^{(p,q)}(F) = 0$ *when* ${}^{L}\tau_{f_i}(m,q) = 0$,

(iii) ${}^{L}\bar{\tau}_G^{(p,q)}(F) = \infty$ *when* ${}^{L}\bar{\tau}_{f_i}(m,q) = \infty$ and

(iv) ${}^{L}\tau_G^{(p,q)}(F) = \infty$ *when* ${}^{L}\tau_{f_i}(m,q) = \infty$ .

Now we state the following two theorems in the line of Theorem 1 and with the help of Theorem A the proof of the theorems can easily be derived:

**Theorem 5** Considering the primary assumptions the following result can be drawn:

$$ max\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\} \le {}^{L}\sigma_G^{(p,q)}(F) $$

$$ \le min\left\{\left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}\right\} $$

and

$$ \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}} \le {}^{L}\bar{\sigma}_G^{(p,q)}(F) \le min\left\{\begin{array}{l}\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}, \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}, \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \\ \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}, \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}, \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}\end{array}\right\}. $$

**Proof**

Now from (3) and in view of (25), we get for a sequence of values of r tending to infinity that

$$ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q) - \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right] $$

i.e., $$ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}} $$

i.e., $$ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)] \ge \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}} \left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}} $$

i.e., $$ \frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}} \ge \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. $$

Since in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)} \ge {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$ \limsup_{r\to\infty} \frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \ge \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}} $$

i.e, $$ {}^{L}\sigma_G^{(p,q)}(F) \ge \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \qquad (29) $$

Analogously from (3) and in view of (28), it follows for a sequence of values of r tending to infinity that

$$ log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \ge log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q) - \epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right] $$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \geq \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}} \geq \left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$.

Now in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)} \geq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \geq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,}\quad {}^{L}\sigma_G^{(p,q)}(F) \geq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \tag{30}$$

Again in view of (26), we have from (21) for all sufficiently large values of r that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}} \leq \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$.

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)} \leq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \leq \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,}\quad {}^{L}\sigma_G^{(p,q)}(F) \leq \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \tag{31}$$

Again

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \leq \left[\frac{\left({}^{L}\sigma_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$.

Since in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)} \leq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\leq\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e,}\qquad {}^{L}\sigma_G^{(p,q)}(F)\leq\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \qquad (32)$$

Further in view of (6), we have from (21) for all sufficiently large values of r that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

$$\text{i.e., } log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e., } log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\leq\left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e., }\quad\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}}\leq\left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$$

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}\leq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\leq\left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e,}\qquad {}^{L}\sigma_G^{(p,q)}(F)\leq\left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \qquad (33)$$

Thus the first part of the theorem follows from (29),(30),(31),(32) and (33).
Now from (3) and in view of (25), we get for all sufficiently large values of r that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\geq log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}\right]$$

$$\text{i.e., } log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\geq log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., } log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)\geq\left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., }\quad\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}}\geq\left[\frac{\left({}^{L}\bar{\sigma}_{f_i}(m,q)-\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)+\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}.$$

Now in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}\geq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\geq\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,}\qquad {}^{L}\bar{\sigma}_G^{(p,q)}(F)\geq\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \qquad (34)$$

Since in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}\leq {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary we get from (13) that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}}\leq\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e,} \qquad {}^{L}\bar{\sigma}_{G}^{(p,q)}(F) \leq \left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$$

Now in view of Theorem A, we get that $\frac{{}^{L}\rho_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)} \leq {}^{L}\rho_{G}^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{G}^{(p,q)}(F)}} \leq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,} \qquad {}^{L}\bar{\sigma}_{G}^{(p,q)}(F) \leq \left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \qquad (36)$$

Also in view of (27), we get from (21) for a sequence of values of r tending to infinity that

$$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \log^{[p-1]}M_{g_i}{}^{-1}\left[\exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

$$\text{i.e., } \log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \log^{[p-1]}\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\tau}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., } \log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., } \frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}} \leq \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\tau}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}.$$

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)} \leq {}^{L}\rho_{G}^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{G}^{(p,q)}(F)}} \leq \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,} \qquad {}^{L}\bar{\sigma}_{G}^{(p,q)}(F) \leq \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \qquad (37)$$

Also in view of (27), we get from (21) for a sequence of values of r tending to infinity that

$$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \log^{[p-1]}M_{g_i}{}^{-1}\left[\exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

$$\text{i.e., } \log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \log^{[p-1]}\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., } \log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \left[\frac{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e., } \frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)}}} \leq \left[\frac{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\tau_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}.$$

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\lambda_{g_i}(m,p)} \leq {}^{L}\rho_{G}^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_{G}^{(p,q)}(F)}} \leq \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}$$

$$\text{i.e,} \qquad {}^{L}\bar{\sigma}_{G}^{(p,q)}(F) \leq \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,p)}}. \qquad (38)$$

Also in view of (27), we get from (21) for a sequence of values of r tending to infinity that

$$\log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \leq \log^{[p-1]}M_{g_i}{}^{-1}\left[\exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(\log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \le \left[\frac{\left({}^{L}\bar{\tau}_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\sigma_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)} \le {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \le \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e,}\quad {}^{L}\bar{\sigma}_G^{(p,q)}(F) \le \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{39}$$

Also in view of (24), we get from (6) for a sequence of values of r tending to infinity that

$$log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}M_{g_i}{}^{-1}\left[exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}\right]$$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le log^{[p-1]}exp^{[p-1]}\left(\frac{log^{[m-1]}exp^{[m-1]}\left\{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)\left(\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\lambda_{f_i}(m,q)}\right)\right\}}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right)^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r) \le \left[\frac{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}$

i.e., $\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)}}} \le \left[\frac{\left({}^{L}\tau_{f_i}(m,q)+\epsilon\right)}{\left({}^{L}\bar{\sigma}_{g_i}(m,p)-\epsilon\right)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}.$

Now in view of Theorem A, we get that $\frac{{}^{L}\lambda_{f_i}(m,q)}{{}^{L}\rho_{g_i}(m,p)} \le {}^{L}\rho_G^{(p,q)}(F)$ and as ε(>0) is arbitrary therefore it follows from above that

$$\liminf_{r\to\infty}\frac{log^{[p-1]}M_{g_i}{}^{-1}M_{f_i}(r)}{\left(log^{[q-1]}(rL(r))\right)^{{}^{L}\rho_G^{(p,q)}(F)}} \le \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}$$

$$\text{i.e,}\quad {}^{L}\bar{\sigma}_G^{(p,q)}(F) \le \left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,p)}}. \tag{40}$$

Therefore the second part of the theorem follows from (34),(35),(36),(37),(38),(39) and(40).

**Theorem 6** Under the primary assumptions as laid down in Theorem 1 the following conclusion can be derived:

$$max\left\{\begin{array}{l}\left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}},\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}},\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}\\ ,\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}},\left[\frac{{}^{L}\sigma_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}},\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\tau_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\end{array}\right\} \le {}^{L}\bar{\tau}_G^{(p,q)}(F) \le \left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}$$

and

$$max\left\{\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}},\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}},\left[\frac{{}^{L}\bar{\sigma}_{f_i}(m,q)}{{}^{L}\bar{\tau}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\lambda_{g_i}(m,q)}}\right\} \le {}^{L}\tau_G^{(p,q)}(F)$$
$$\le min\left\{\left[\frac{{}^{L}\tau_{f_i}(m,q)}{{}^{L}\bar{\sigma}_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}},\left[\frac{{}^{L}\bar{\tau}_{f_i}(m,q)}{{}^{L}\sigma_{g_i}(m,p)}\right]^{\frac{1}{{}^{L}\rho_{g_i}(m,q)}}\right\}.$$

**Conclusion.**

The prime moto of our work as presented in the paper is to tackle the estimation of different relative growth indicators of higher index in case of entire algebroidal functions defined in the finite complex plane using the techniques and properties of slowly changing functions.But one may raise question about the effect of deducing the identical class of theorems for the functions analytic in the unit disc and these deductions are still open and virgin in the field of comparative growth estimations of analytic functions.

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